

Spectra of formulas with bounded quantifier alternations ¹A.D. Matushkin, M.E. Zhukovskii ²**Abstract**

Spectrum of a first order sentence is the set of all α such that $G(n, n^{-\alpha})$ does not obey zero-one law w.r.t. this sentence. We have proved that the minimal number of quantifier alternations of a first order sentence with infinite spectrum equals 3.

1 Previous results on zero-one laws

In this paper, we consider first order sentences about graphs (a signature consists of two predicates \sim (adjacency) and $=$ (equality) of arity 2) [1, 2]. Recall that a *quantifier depth* $q(\phi)$ of a formula ϕ is the number of quantifiers in the longest past of nested quantifiers in this formula. Let $G(n, p)$ be a binomial random graph [3, 4] with n vertices and the probability p of appearing of an edge. We say that $G(n, p)$ *obeys zero-one law w.r.t. a first order sentence* ϕ , if either a.a.s. (asymptotically almost surely) $G(n, p) \models \phi$, or a.a.s. $G(n, p) \models \neg(\phi)$.

Let $S(\phi)$ be the set of all $\alpha > 0$ such that $G(n, n^{-\alpha})$ does not obey zero-one law w.r.t. ϕ . This set is called a *spectrum of ϕ* . In 1988 [5], S. Shelah and J. Spencer proved that there are only rational numbers in $S(\phi)$ for any first order sentence ϕ . In 1991 [6], J. Spencer proved that there exists first order sentence with an infinite spectrum and the quantifier depth 14. In 2012 [7], M. Zhukovskii proved that, for any first order sentence ϕ with the quantifier depth 3, $S(\phi) \cap (0, 1) = \emptyset$. Moreover, for any first order sentence ϕ with the quantifier depth 4, $S(\phi) \cap (0, 1/2) = \emptyset$. Later [8], it was proved that, for any first order sentence ϕ , the set $S(\phi) \cap (1, \infty)$ is finite. In [9], a first order sentence with the quantifier depth 5 and an infinite spectrum was obtained. This formula is given in the statement below.

Theorem 1 *Let $m \in \mathbb{N}$, $\alpha = \frac{1}{2} + \frac{1}{2(m+1)}$ and $p = n^{-\alpha}$. Then the random graph $G(n, p)$ does not obey zero-one law w.r.t. the sentence*

$$\phi = \exists x_1 \exists x_2 \left[\left(\exists x_3 \exists x_4 \left(\bigwedge_{1 \leq i < j \leq 4} (x_i \sim x_j) \right) \right) \wedge (\varphi(x_1, x_2)) \right],$$

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where

$$\begin{aligned} \varphi(x_1, x_2) = & \forall y_1 ([y_1 \sim x_1] \vee [y_1 \sim x_2] \vee [\forall y_2 (\neg[(y_2 \sim x_1) \wedge (y_2 \sim y_1)])]) \vee \\ & [\exists z (z \sim x_1) \wedge (z \sim x_2) \wedge (\forall u ([\neg[(u \sim z) \wedge (u \sim y_1)]) \vee (u \sim x_1) \vee (u \sim x_2)])]). \end{aligned}$$

So, a minimal quantifier depth of a first order sentence with an infinite spectrum equal either 4, or 5.

Note that the maximal number of quantifier alternations over all sequences of nested quantifiers in ϕ equals 3 (we call this value the *number of quantifier alternations of ϕ*). It is essential that all the negations are applied to atomic formulas only. A prenex normal form of ϕ with the quantifier depth 8 is given below

$$\tilde{\phi} = \exists x_1 \exists x_2 \exists x_3 \exists x_4 \forall y_1 \forall y_2 \exists z \forall u \left[\left(\bigwedge_{1 \leq i < j \leq 4} (x_i \sim x_j) \right) \wedge (\tilde{\varphi}(x_1, x_2, y_1, y_2, z, u)) \right], \quad (1)$$

where

$$\begin{aligned} \tilde{\varphi}(x_1, x_2, y_1, y_2, z, u) = & [y_1 \sim x_1] \vee [y_1 \sim x_2] \vee [\neg((y_2 \sim x_1) \wedge (y_2 \sim y_1))] \vee \\ & [(z \sim x_1) \wedge (z \sim x_2) \wedge (\neg[(u \sim z) \wedge (u \sim y_1)])] \vee [u \sim x_1] \vee [u \sim x_2]. \end{aligned}$$

This raises the following questions.

1. What is the minimal quantifier depth of a first order sentence with an infinite spectrum, 4 or 5?
2. What is the minimal number of quantifier alternations of a first order sentence with an infinite spectrum, 3 or less?
3. What is the minimal quantifier depth of a first order sentence in a prenex normal form with an infinite spectrum, 4, 5, 6, 7 or 8?

We partially answer these questions in Sections 4, 5.

2 Existence and extension statements

Let ϕ be a first order sentence in a prenex normal form. We call ϕ an *existence sentence*, if all quantifiers of ϕ equal \exists . We call ϕ an *extension sentence*, if the sequence of all quantifiers of ϕ equals $\forall \dots \forall \exists \dots \exists$. We say that an existence sentence expresses an existence property,

and an extension sentence expresses an extension property. An asymptotical behavior of probabilities of the random graph existence and extension properties was widely studied in [10, 11, 12, 13]. We summarize this study in the result given below.

Let G, H be two graphs such that $H \subset G$, $V(H) = \{a_1, \dots, a_s\}$, $V(G) \setminus V(H) = \{b_1, \dots, b_m\}$, $s, m \geq 1$. Let $\rho(H)$ be a maximal fraction $e(Q)/v(Q)$ over all subgraphs $Q \subset H$ ($\rho(H)$ is called *the maximal density* of H). Here $e(Q), v(Q)$ denote the numbers of edges and vertices in Q respectively. Let $\rho(G, H)$ be a maximal fraction $(e(Q) - e(H))/(v(Q) - v(H))$ over all Q such that $H \subset Q \subset G$. We say that a graph has *the (G, H) -extension property*, if, for any its distinct vertices y_1, \dots, y_s , there exist distinct vertices x_1, \dots, x_m such that, for all $i \in \{1, \dots, s\}$, $j \in \{1, \dots, m\}$, $y_i \neq x_j$ and the adjacency relation $a_i \sim b_j$ implies the adjacency relation $y_i \sim x_j$.

Theorem 2 *Let $\rho(H) \neq 0$. If $p \gg n^{-1/\rho(H)}$, then a.a.s. in $G(n, p)$ there is an induced copy of H . If $p \ll n^{-1/\rho(H)}$, then a.a.s. in $G(n, p)$ there is no copy of H .*

Let $\rho(G, H) \neq 0$. If $p \gg n^{-1/\rho(G, H)}$, then a.a.s. $G(n, p)$ has the (G, H) -extension property. If $p \ll n^{-1/\rho(G, H)}$, then a.a.s. $G(n, p)$ does not have the (G, H) -extension property.

It is not difficult to see that Theorem 2 implies finiteness of spectra of all existence and extension sentences (see Section 4).

The next step is to consider sentences in prenex normal form that have 2 alternations. We call ϕ a *double-extension sentence*, if the sequence of all quantifiers of ϕ equals $\forall \dots \forall \exists \dots \exists \forall \dots \forall$ (the respective properties are called *double-extension* as well). An asymptotical behavior of probabilities of the random graph double-extension properties was studied in [14, 15].

Let W, G, H be three graphs such that $H \subset G \subset W$, $V(H) = \{a_1, \dots, a_s\}$, $V(G) \setminus V(H) = \{b_1, \dots, b_m\}$, $V(W) \setminus V(G) = \{c_1, \dots, c_r\}$, $s \geq 0$, $r, m \geq 1$. Assume that in W there are edges between each connected component of $W|_{\{c_1, \dots, c_r\}}$ and $W|_{\{b_1, \dots, b_m\}}$. Let \mathcal{W} be a finite set of graphs such that all $W \in \mathcal{W}$ satisfy the above conditions (but r depends on W). We say that a graph has *the (\mathcal{W}, G, H) -double-extension property*, if, for any its distinct vertices y_1, \dots, y_s , there exist distinct vertices x_1, \dots, x_m such that, for all $W \in \mathcal{W}$ and all distinct vertices $z_1, \dots, z_{r(W)}$,

- for all $i \in \{1, \dots, s\}$, $j \in \{1, \dots, m\}$, $y_i \neq x_j$ and the adjacency relation $a_i \sim b_j$ implies the adjacency relation $y_i \sim x_j$,
- either there exists $h \in \{1, \dots, r(W)\}$ and $i \in \{1, \dots, s\}$ such that $[z_h = y_i] \vee [(z_h \approx y_i) \wedge (c_h \sim a_i)]$,
or there exist $h \in \{1, \dots, r(W)\}$ and $j \in \{1, \dots, m\}$ such that $[z_h = x_j] \vee [(z_h \approx x_j) \wedge (c_h \sim b_j)]$.

Theorem 3 *Let, for all $W \in \mathcal{W}$, $\rho(W, G) > \rho(G, H) > 0$ and $n^{-1/\rho(W, G)} \gg p \gg n^{-1/\rho(G, H)}$. Then a.a.s. $G(n, p)$ has the (\mathcal{W}, G, H) -double-extension property.*

We have proved that Theorems 2, 3 imply the finiteness of spectra of double-extension sentences (see Section 4 as well).

So, we generalize the well-known results about existence, extension and double-extension properties and prove that spectra of all first order sentences with at most 2 quantifier alternations are finite.

3 Logical preliminaries

3.1 Some notations

Recall that a *rooted tree* T_R is a tree with one distinguished vertex R , which is called *the root*. If R, \dots, x, y is a simple path in T_R , then x is called a parent of y and y is called a child of x . The relation of being a *descendant* is the transitive and reflexive closure of the relation of being a child. If $v \in V(T_R)$, then $T_R[v]$ denotes the subforest of T_R spanned by the set of all descendants of v (children of v are its roots).

For two first order formulas $\phi_1(x_1, \dots, x_s), \phi_2(y_1, \dots, y_s)$ ($s \in \{0, 1, 2, \dots\}$), we say that they are (asymptotically) *equivalent* (and write $\varphi_1 \cong \varphi_2$), if there exists $n \in \mathbb{N}$ such that for any graph G on at least n vertices and any its vertices v_1, \dots, v_s either $G \models (\phi_1(v_1, \dots, v_s)) \wedge (\phi_2(v_1, \dots, v_s))$, or $G \models (\neg(\phi_1(v_1, \dots, v_s)) \wedge (\neg(\phi_2(v_1, \dots, v_s))))$. We say that a set of graphs C is a (asymptotical) *first order property of a graph*, if there exists a first order sentence ϕ and $n \in \mathbb{N}$ such that, for any G on at least n vertices, $G \in C$ if and only if $G \models \phi$ (in this case, we say that ϕ *expresses* C).

3.2 Language \mathcal{F}

It is easy to see that any first order formula (not necessarily sentence) is equivalent to a formula constructed of the following symbols: variables, relational symbols $\sim, =, \neq, \neq$, conjunctions \wedge , disjunctions \vee and quantifiers \forall, \exists . We denote the set of formulas in this language by \mathcal{F} .

Let us state a simple observation of formulas in \mathcal{F} .

Lemma 1 *Let $Z \in \{\wedge, \vee\}$, $z_1, z_2 \in \{\forall, \exists\}$. Then, for any two formulas $\varphi_1(x_1, \dots, x_s), \varphi_2(x_1, \dots, x_m) \in \mathcal{F}$ (not necessarily with s and m free variables respectively),*

$$[z_1 x_1 \dots z_1 x_s (\varphi_1(x_1, \dots, x_s))] Z [z_2 x_1 \dots z_2 x_m (\varphi_2(x_1, \dots, x_m))] \cong$$

$$z_1x_1 \dots z_1x_s z_2x_{s+1} \dots z_2x_{s+m} ([\varphi_1(x_1, \dots, x_s)]Z[\varphi_2(x_{s+1}, \dots, x_{s+m})]).$$

For a formula $\phi \in \mathcal{F}$, define its *nesting forest* $F(\phi)$ in the following way.

- If ϕ is an atomic formula, then its nesting forest is an empty graph.
- Consider a formula $\varphi(x)$. If it has an empty nesting forest, then the nesting forest of the formula $\phi = \exists x(\varphi(x))$ (the formula $\phi = \forall x(\varphi(x))$) is an isolated vertex labeled by \exists (by \forall). This vertex is a trivial tree rooted in its only vertex. Otherwise, let $F(\varphi(x)) = T_{t_1}^1 \sqcup \dots \sqcup T_{t_m}^m$, where $T_{t_1}^1, \dots, T_{t_m}^m$ are trees rooted in t_1, \dots, t_m respectively. Then the nesting forest of the formula $\phi = \exists x(\varphi(x))$ (the formula $\phi = \forall x(\varphi(x))$) is a tree obtained by adding a vertex t (which is the root of this tree) labeled by \exists (by \forall) to $F(\varphi(x))$ and edges from t to each of t_1, \dots, t_m .
- If $\phi = (\varphi_1) \wedge (\varphi_2)$ (or $\phi = (\varphi_1) \vee (\varphi_2)$), then $F(\phi)$ is the disjoint union of $F(\varphi_1)$, $F(\varphi_2)$.

Consider a formula $\phi(x_1, \dots, x_s) \in \mathcal{F}$ and its nesting forest $F(\phi) = T_{t_1}^1 \sqcup \dots \sqcup T_{t_m}^m$ consisting of trees $T_{t_1}^1, \dots, T_{t_m}^m$ rooted in t_1, \dots, t_m respectively. Let v be a vertex of $T_{t_i}^i$ for some $i \in \{1, \dots, m\}$. Consider the forest $T_{t_i}^i[v]$. Let V be the set of all vertices of $T_{t_i}^i$ such that v is a descendant for each of them, $[V] := V \cup \{v\}$. Obviously, $T_{t_i}^i|_{[V]}$ is the path $t_i \dots v$. Each of the vertices of this path corresponds to a bound variable of ϕ . Let y_1, \dots, y_r be these variables (y_{i+1}, y_i corresponds to a child and a parent respectively). Then $T_{t_i}^i[v]$ is the nested forest of a subformula $\varphi(x_1, \dots, x_s, y_1, \dots, y_r)$ of ϕ . The formula $\varphi(x_1, \dots, x_s, y_1, \dots, y_r)$ is called a *nested subformula* of ϕ , the forest $F(\varphi(x_1, \dots, x_s, y_1, \dots, y_r))$ is called a *nested subforest* of $F(\phi)$.

Note that the quantifier depth of ϕ is the length of the longest path starting in a root (we denote it by $q(\phi)$). For a path in $F(\phi)$ starting in a root consider the number of labels alternations (the number of (unordered) pairs of neighbors $\forall\exists$ and $\exists\forall$). For example, the number of labels alternations of the path $\exists\forall\forall\exists\exists\forall$ equals 3. The maximal number of labels alternations over all paths starting in roots of $F(\phi)$ is called the *number of quantifier alternations* of ϕ (we denote it by $ch(\phi)$).

3.3 Normal forms

A formula $\phi \in \mathcal{F}$ is in *prenex normal form* (PNF) (we also say that ϕ is a *PNF formula* or a *PNF sentence*), if $F(\phi)$ is a path (all quantifiers are in the beginning of the formula). We say that $\hat{\phi}$ is a *PNF of ϕ* , if $\hat{\phi} \in \mathcal{F}$, $\hat{\phi}$ is in PNF and $\hat{\phi} \cong \phi$. It is known [16], that for

any first order formula (which is not necessarily in \mathcal{F}) there exists an equivalent first order formula in PNF. This immediately implies that ϕ has a PNF.

The formula ϕ is in *no-equivalence prenex normal form* (NEPNF) (we also say that ϕ is NEPNF *formula* or a NEPNF *sentence*), if ϕ is in PNF, and is constructed as follows. Consider an arbitrary sequence $z = (z_1, \dots, z_m)$ of symbols from $\{\forall, \exists\}$. Let a formula $\phi_1(x_1, \dots, x_m) \in \mathcal{F}$ has no quantifiers and no relations $=$ and \neq . For each $j \in \{1, \dots, m-1\}$ a formula $\phi_{j+1}(x_1, \dots, x_m)$ is obtained from $\phi_j(x_1, \dots, x_m)$ in the following way:

$$\phi_{j+1}(x_1, \dots, x_m) = (x_{j+1} \neq x_j) \wedge \dots \wedge (x_{j+1} \neq x_1) \wedge (\phi_j(x_1, \dots, x_m)), \quad \text{if } z_{j+1} = \exists,$$

$$\phi_{j+1}(x_1, \dots, x_m) = (x_{j+1} = x_j) \vee \dots \vee (x_{j+1} = x_1) \vee (\phi_j(x_1, \dots, x_m)), \quad \text{if } z_{j+1} = \forall.$$

Finally, $\phi = z_1 x_1 \dots z_m x_m (\phi_m(x_1, \dots, x_m))$. We say that $(\phi_1(x_1, \dots, x_m), z)$ is NE-basis of ϕ .

Lemma 2 *For any PNF sentence $\phi \in \mathcal{F}$ there exists an NEPNF sentence $\hat{\phi} \in \mathcal{F}$ with the same sequence of quantifiers such that $\phi \cong \hat{\phi}$.*

Proof. Let $\phi = z_1 x_1 \dots z_m x_m (\varphi(x_1, \dots, x_m))$, where z_1, \dots, z_m is a sequence of symbols from $\{\forall, \exists\}$. Set $\hat{\phi}_1(x_1, \dots, x_m) = \varphi(x_1, \dots, x_m)$. For each $j \in \{1, \dots, m-1\}$ a formula $\hat{\phi}_{j+1}(x_1, \dots, x_m)$ is obtained from $\hat{\phi}_j(x_1, \dots, x_m)$ in the following way.

First, $\hat{\phi}_{j+1}^0(x_1, \dots, x_m)$ is obtained from $\hat{\phi}_j(x_1, \dots, x_m)$ by assuming that all

$$x_1 = x_{j+1}, \dots, x_j = x_{j+1}$$

are false, and all

$$x_1 \neq x_{j+1}, \dots, x_j \neq x_{j+1}$$

are true. For any $i \in \{1, \dots, j+1\}$, $\hat{\phi}_j^i(x_1, \dots, x_m)$ is obtained from $\hat{\phi}_j(x_1, \dots, x_m)$ by assuming that all

$$x_1 = x_{j+1}, \dots, x_{i-1} = x_{j+1}, x_i \neq x_{j+1}, x_{i+1} = x_{j+1}, \dots, x_j = x_{j+1}$$

are false, and all

$$x_1 \neq x_{j+1}, \dots, x_{i-1} \neq x_{j+1}, x_i = x_{j+1}, x_{i+1} \neq x_{j+1}, \dots, x_j \neq x_{j+1}$$

are true.

Second, if $z_{j+1} = \exists$, then

$$\hat{\phi}_{j+1}(x_1, \dots, x_m) = \hat{\phi}_j^0(x_1, \dots, x_m) \vee \left[\bigvee_{i=1}^j (\hat{\phi}_j^i(x_1, \dots, x_j, x_i, x_{j+2}, \dots, x_m)) \right].$$

Otherwise,

$$\hat{\phi}_{j+1}(x_1, \dots, x_m) = \hat{\phi}_j^0(x_1, \dots, x_m) \wedge \left[\bigwedge_{i=1}^j (\hat{\phi}_j^i(x_1, \dots, x_j, x_i, x_{j+2}, \dots, x_m)) \right].$$

Let $\hat{\phi}$ be the NEPNF formula with the NE-basis $(\hat{\phi}_m(x_1, \dots, x_m), (z_1, \dots, z_m))$. It is easy to see that $\phi \cong \hat{\phi}$. Both formulas have the same sequence of quantifiers. \square

We will frequently use the following corollary.

Lemma 3 *Let $\phi = \exists x (\varphi(x)) \in \mathcal{F}$. Then there exists an NEPNF formula $\hat{\phi} = \exists x (\hat{\varphi}(x)) \in \mathcal{F}$ such that $\phi \cong \hat{\phi}$ and $\text{ch}(\phi) = \text{ch}(\hat{\phi})$.*

Proof. Let F be a nesting forest of a formula with the quantifier depth q . Moreover, let F be a rooted tree with a root $t(F)$. Denote by $t_1^r(F), \dots, t_{a(r,F)}^r(F)$ all the vertices of F which are at the distance $r - 1$ from $t(F)$, where $r \in \{1, \dots, q\}$, $a(r, F) \in \{1, 2, \dots\}$. Obviously, $a(1, F) = 1$, $a(r, F) \geq 1$ for all $r \in \{2, \dots, q\}$. Let r be the first positive integer such that $a(r, F) > 1$ (if there is no such r , then set $r = q + 1$). Let

$$\mu[F] = q + 1 - r.$$

Note that if F is a simple path with an end-point $t(F)$, then $\mu[F] = 0$.

Consider a sentence $\phi = \exists x (\varphi(x)) \in \mathcal{F}$ such that $\text{ch}(\phi) = k$. By Lemma 2, it is sufficient to prove that there exists a PNF sentence $\hat{\phi} = \exists x (\hat{\varphi}(x)) \in \mathcal{F}$ such that $\phi \cong \hat{\phi}$ and $\text{ch}(\hat{\phi}) = k$. If $\mu[F(\phi)] = 0$, then we are done ($\hat{\phi} = \phi$). Suppose that $\mu[F(\phi)] = m \in \mathbb{N}$, and that for any formula ζ (not necessarily closed and with an arbitrary first quantifier) with $\mu[F(\zeta)] < m$ the existence of an equivalent PNF sentence with the same number of quantifier alternations and the same first quantifier is already proven.

Let $\phi = z_1 x_1 \dots z_s x_s (\varphi(x_1, \dots, x_s))$, where $s = q - \mu[F(\phi)]$, $z_1 = \exists$, $z_2, \dots, z_s \in \{\forall, \exists\}$, and the first symbol of $\varphi(x_1, \dots, x_s)$ is not a quantifier. The formula $\varphi(x_1, \dots, x_s)$ is a logical combination L (disjunctions and conjunctions) of formulas

$$\exists x (\hat{\varphi}_i(x_1, \dots, x_s, x)), \quad \forall x (\hat{\varphi}^j(x_1, \dots, x_s, x)).$$

Let $I = \{1, \dots, |I|\}$ be the set of all such i s and $J = \{1, \dots, |J|\}$ be the set of all such j s. So,

$$\varphi(x_1, \dots, x_s) = L(\exists x (\hat{\varphi}_i(x_1, \dots, x_s, x)), i \in I; \forall x (\hat{\varphi}^j(x_1, \dots, x_s, x)), j \in J).$$

Obviously, for all $i \in I, j \in J$,

$$\mu[F(\hat{\varphi}_i(x_1, \dots, x_s, x))] < m, \quad \mu[F(\hat{\varphi}^j(x_1, \dots, x_s, x))] < m.$$

By the induction hypothesis, for all $i \in I, j \in J$ there exist PNF formulas

$$\exists x (\tilde{\varphi}_i(x_1, \dots, x_s, x)) \cong \exists x (\hat{\varphi}_i(x_1, \dots, x_s, x)),$$

$$\forall x (\tilde{\varphi}^j(x_1, \dots, x_s, x)) \cong \forall x (\hat{\varphi}^j(x_1, \dots, x_s, x)),$$

such that

$$\text{ch}(\exists x (\tilde{\varphi}_i(x_1, \dots, x_s, x))) = \text{ch}(\exists x (\hat{\varphi}_i(x_1, \dots, x_s, x))),$$

$$\text{ch}(\forall x (\tilde{\varphi}^j(x_1, \dots, x_s, x))) = \text{ch}(\forall x (\hat{\varphi}^j(x_1, \dots, x_s, x))).$$

Let

$$\tilde{\psi}(x_1, \dots, x_s) = L(\exists x (\tilde{\varphi}_i(x_1, \dots, x_s, x)), i \in I; \forall x (\tilde{\varphi}^j(x_1, \dots, x_s, x)), j \in J).$$

Then the formulas ϕ and

$$\psi = z_1 x_1 \dots z_s x_s (\tilde{\psi}(x_1, \dots, x_s))$$

are equivalent and have the same numbers of quantifier alternations. Moreover, $F(\psi)$ is a rooted tree with exactly one vertex with a degree greater than 2. The distance between this vertex and the root $t(F(\psi))$ is $s - 1$. Let the distance between this vertex and a vertex with the biggest distance from the root equal r . Let us construct a formula $\psi^0 \cong \psi$ such that $\text{ch}(\psi^0) = \text{ch}(\psi)$, $F(\psi^0)$ is a rooted tree with at most one vertex with a degree greater than 2, and the distance between this vertex (if it exists) and a vertex with the biggest distance from the root is less than r . Obviously, we get the target formula $\hat{\phi}$ after applying such a construction at most r times.

For all $i \in I, j \in J$ let us find positive integers d_i, d^j such that

$$\exists x^1 (\tilde{\varphi}_i(x_1, \dots, x_s, x^1)) = \exists x^1 \dots \exists x^{d_i} (\tilde{\psi}_i(x_1, \dots, x_s, x^1, \dots, x^{d_i})),$$

$$\forall x^1 (\tilde{\varphi}^j(x_1, \dots, x_s, x^1)) = \forall x^1 \dots \forall x^{d^j} (\tilde{\psi}^j(x_1, \dots, x_s, x^1, \dots, x^{d^j})),$$

where the formulas $\tilde{\psi}_i(x_1, \dots, x_s, x^1, \dots, x^{d_i})$, $\tilde{\psi}^j(x_1, \dots, x_s, x^1, \dots, x^{d^j})$ either have no quantifiers, or \forall, \exists are the quantifier symbols they begin from respectively. Set $D_I = \sum_{i \in I} d_i$, $D_J = \sum_{j \in J} d^j$. Without loss of generality, assume $z_s = \exists$.

By Lemma 1, there exists a formula (if $z_s = \forall$, then this formula starts with \forall)

$$\tilde{\psi}^0(x_1, \dots, x_s) =$$

$$\exists x_{s+1} \dots \exists x_{s+D_I} \forall x_{s+D_I+1} \dots \forall x_{s+D_I+D_J} (\hat{\psi}(x_1, \dots, x_{s+D_I+D_J})) \cong \tilde{\psi}(x_1, \dots, x_s)$$

such that

$$\text{ch}(z_1x_1 \dots z_sx_s (\tilde{\psi}^0(x_1, \dots, x_s))) = \text{ch}(z_1x_1 \dots z_sx_s (\tilde{\psi}(x_1, \dots, x_s))).$$

Moreover, $F(z_1x_1 \dots z_sx_s (\tilde{\psi}^0(x_1, \dots, x_s)))$ is a tree with exactly one vertex with a degree greater than 2, and the distance between this vertex and a vertex with the biggest distance from the root is less than r . Finally, set

$$\psi^0 = z_1x_1 \dots z_sx_s (\tilde{\psi}^0(x_1, \dots, x_s)). \quad \square$$

3.4 Ehrenfeucht games

We consider three modification of Ehrenfeucht game.

1. The game $\text{EHR}(G, H, q)$ is played on graphs G and H . There are two players (Spoiler and Duplicator) and a fixed number of rounds q . At the ν -th round ($1 \leq \nu \leq q$), Spoiler chooses either a vertex x_ν of G or a vertex y_ν of H (which does not coincide with any of chosen vertices). Duplicator chooses a vertex of the other graph (which does not coincide with any of chosen vertices as well). At the end of the game, the distinct vertices x_1, \dots, x_q of G , y_1, \dots, y_q of H are chosen. Duplicator wins if and only if the map $f(x_i) = y_i$, $i \in \{1, \dots, q\}$, is an isomorphism of $G|_{\{x_1, \dots, x_q\}}$ and $H|_{\{y_1, \dots, y_q\}}$.
2. In the game $\text{EHR}(G, H, q, \leq k)$, there are q rounds as well. The only difference with the game $\text{EHR}(G, H, q)$ is that Spoiler can alternate at most k times (if in the i -th round Spoiler chooses a vertex, say, in G , and in the $i + 1$ -th round — in H (or vice versa), then we say that he *alternates*).
3. The most strict rules (for Spoiler) are in the game $\text{EHR}(G, H, q, k)$. The only difference with the game $\text{EHR}(G, H, q, \leq k)$ is that Spoiler must alternates exactly k times.

Our results on first order properties of random graphs are based on the following typical arguments on the connection between an elementary equivalence and Ehrenfeucht game.

Lemma 4 *The following two properties are equivalent:*

- 1) *Spoiler has a winning strategy in $\text{EHR}(G, H, q)$;*
- 2) *there is $\phi \in \mathcal{F}$ with $\text{q}(\phi) = q$ such that $G \models \phi$, $H \models \neg(\phi)$.*

This statement is a particular case of Ehrenfeucht theorem [17].

The next two lemmas have typical proofs. We give it here for the sake of convenience.

Lemma 5 *The following two properties are equivalent:*

- 1) *Spoiler has a winning strategy in $\text{EHR}(G, H, q, \leq k)$;*
- 2) *there is $\phi \in \mathcal{F}$ with $q(\phi) = q$ and $\text{ch}(\phi) \leq k$ such that $G \models \phi$, $H \models \neg(\phi)$.*

Proof. First, let us prove that 2) implies 1). Let $\phi \in \mathcal{F}$ be a sentence such that $\text{ch}(\phi) \leq k$, $q(\phi) = q$, $G \models \phi$ and $H \models \neg(\phi)$. We will describe a winning strategy of Spoiler by an induction on the number of played rounds. The sentence ϕ is a logical combination (disjunctions and conjunctions) of sentences $\varphi_i = \exists x (\hat{\varphi}_i(x))$ and $\varphi^j = \forall x (\hat{\varphi}^j(x))$. Obviously, one of these sentences is true for G and not true for H . Let β_1 be the root of the nesting forest (tree) of this sentence. If, say,

$$G \models \exists x (\hat{\varphi}_1(x)), \quad H \models \neg(\exists x (\hat{\varphi}_1(x))),$$

then set $\varphi_1(x) := \hat{\varphi}_1(x)$. Spoiler in the **first round** chooses a vertex v_1 such that $G \models \varphi_1(v_1)$. Duplicator chooses a vertex u_1 . Obviously, $H \models \neg(\varphi_1(u_1))$. Denote the root of $F(\phi)$ by β_1 . If, say,

$$G \models \forall x (\hat{\varphi}^1(x)), \quad H \models \neg(\forall x (\hat{\varphi}^1(x))),$$

then set $\varphi_1(x) := \hat{\varphi}^1(x)$. Spoiler in the first round chooses a vertex u_1 such that $H \models \neg(\varphi_1(u_1))$. Duplicator chooses a vertex v_1 . Obviously, $G \models \varphi_1(v_1)$.

Fix $m \in \{2, \dots, k\}$, $\ell = \ell(m-1) \in \{1, \dots, m-1\}$ and vertices v_1, \dots, v_{m-1} , u_1, \dots, u_{m-1} (not necessarily distinct) in the graphs G, H respectively. Suppose that $v_{i_1}, \dots, v_{i_\ell}$, $u_{i_1}, \dots, u_{i_\ell}$ are all distinct vertices of v_1, \dots, v_{m-1} , u_1, \dots, u_{m-1} respectively. Moreover, $v_j = v_{i_r}$ if and only if $u_j = u_{i_r}$. Suppose that ℓ **rounds are played**, and the vertices $v_{i_1}, \dots, v_{i_\ell}$, $u_{i_1}, \dots, u_{i_\ell}$ are chosen in the graphs G, H respectively. Moreover, suppose that in ϕ there exists a nested subformula $\varphi_{m-1}(x_1, \dots, x_{m-1})$ such that $q(\varphi_{m-1}(x_1, \dots, x_{m-1})) = q - m + 1$,

$$G \models \varphi_{m-1}(v_1, \dots, v_{m-1}), \quad H \models \neg(\varphi_{m-1}(u_1, \dots, u_{m-1})).$$

The formula $\varphi_{m-1}(x_1, \dots, x_{m-1})$ is a logical combination (disjunctions and conjunctions) of formulas

$$\exists x_m (\hat{\varphi}_i(x_1, \dots, x_m)), \quad \forall x_m (\hat{\varphi}^j(x_1, \dots, x_m)).$$

Obviously, (at least) one of these formulas is true for G on v_1, \dots, v_{m-1} and not true for H on u_1, \dots, u_{m-1} . Let β_m be the root of the nesting forest of such a formula. If, say,

$$G \models \exists x_m (\hat{\varphi}_1(v_1, \dots, v_{m-1}, x_m)), \quad H \models \neg(\exists x_m (\hat{\varphi}_1(v_1, \dots, v_{m-1}, x_m))),$$

then we find a vertex v_m such that $G \models \hat{\varphi}_1(v_1, \dots, v_{m-1}, v_m)$ and set $\varphi_m(x_1, \dots, x_m) = \hat{\varphi}_1(x_1, \dots, x_m)$. If $v_m \in \{v_1, \dots, v_{m-1}\}$, then Spoiler “skips” this round, and we set $u_m = u_j$, where $j \in \{1, \dots, m-1\}$ is a number such that $v_j = v_m$. Otherwise, Spoiler chooses a vertex $v_{i_{\ell+1}} = v_m$ and Duplicator chooses a vertex $u_{i_{\ell+1}} = u_m$. Obviously, in both cases, $H \models \neg(\varphi_m(u_1, \dots, u_m))$. If, say,

$$G \models \forall x_m (\hat{\varphi}^1(v_1, \dots, v_{m-1}, x_m)), \quad H \models \neg(\forall x_m (\hat{\varphi}^1(v_1, \dots, v_{m-1}, x_m))),$$

then fix a vertex u_m such that $H \models \neg(\hat{\varphi}^1(u_1, \dots, u_{m-1}, u_m))$ and set $\varphi_m(x_1, \dots, x_m) = \hat{\varphi}^1(x_1, \dots, x_m)$. If $u_m \in \{u_1, \dots, u_{m-1}\}$, then Spoiler “skips” this round, and we set $v_m = v_j$, where $j \in \{1, \dots, m-1\}$ is a number such that $u_j = u_m$. Otherwise, Spoiler chooses a vertex $u_{i_{\ell+1}} = u_m$ and Duplicator chooses a vertex $v_{i_{\ell+1}} = v_m$. Obviously, in both cases, $G \models \varphi_m(v_1, \dots, v_m)$.

This strategy is winning for Spoiler in $\text{EHR}(G, H, q)$. Moreover, it is easy to see that Spoiler alternates \tilde{k} times, where $\tilde{k} \leq k$ is the number of labels alternations in the path $\beta_1 \beta_{\ell(2)} \dots \beta_{\ell(q)}$.

It remains to prove that 1) implies 2). Let Spoiler have a winning strategy in the game $\text{EHR}(G, H, k, q)$ with a first move in G . Let us construct a sentence $\phi \in \mathcal{F}$ such that $\text{ch}(\phi) = k$, $\text{q}(\phi) = q$, $G \models \phi$ and $H \models \neg(\phi)$.

Let, after q rounds, distinct vertices v_1, \dots, v_q in G and u_1, \dots, u_q in H be chosen. As Spoiler wins in q rounds, there is a formula $\varphi_q(x_1, \dots, x_q) \in \mathcal{F}$ such that $\text{q}(\varphi_q(x_1, \dots, x_q)) = 0$ and $G \models \varphi_q(v_1, \dots, v_q)$, $H \models \neg(\varphi_q(u_1, \dots, u_q))$.

Fix $m \in \{0, \dots, q-1\}$. Let after m rounds, distinct vertices v_1, \dots, v_m in G and u_1, \dots, u_m in H be chosen. In the $m+1$ -th round, Spoiler chooses, say, a vertex $v_{m+1} \in V(G)$ (according to his winning strategy). Suppose that, for any choice of Duplicator (denote it by u_{m+1}), there is a formula $\varphi_{m+1}^{u_{m+1}}(x_1, \dots, x_{m+1}) \in \mathcal{F}$ such that $\text{q}(\varphi_{m+1}^{u_{m+1}}(x_1, \dots, x_{m+1})) = q - m - 1$ and

$$G \models \varphi_{m+1}^{u_{m+1}}(v_1, \dots, v_{m+1}), \quad H \models \neg(\varphi_{m+1}^{u_{m+1}}(u_1, \dots, u_{m+1})).$$

Note that, for a fixed number of free variables, there is only a finite number of representatives of \cong -equivalence classes of formulas in \mathcal{F} with a fixed quantifier depth (see, e.g., [16]). Therefore, there are a positive constant C (which does not depend on $|V(G)|$, $|V(H)|$) and a set $\mathcal{U} \subset V(H)$ with $|\mathcal{U}| \leq C$ such that the following property holds. For any $u_{m+1} \in V(H)$, there exists $u \in \mathcal{U}$ such that $\varphi_{m+1}^{u_{m+1}}(x_1, \dots, x_{m+1}) \cong \varphi_{m+1}^u(x_1, \dots, x_{m+1})$. Set

$$\varphi_m(x_1, \dots, x_m) = \exists x_{m+1} \left(\bigwedge_{u \in \mathcal{U}} (\varphi_{m+1}^u(x_1, \dots, x_{m+1})) \right).$$

Obviously, $G \models \varphi_m(v_1, \dots, v_m)$ and $H \models \neg(\varphi_m(u_1, \dots, u_m))$.

Finally, let Spoiler choose a vertex $u_{m+1} \in V(H)$ and, for any choice of Duplicator $v_{m+1} \in V(G)$, there exists a formula $\varphi_{m+1}^{v_{m+1}}(x_1, \dots, x_{m+1}) \in \mathcal{F}$ such that $q(\varphi_{m+1}^{v_{m+1}}(x_1, \dots, x_{m+1})) = q - m - 1$ and

$$G \models \varphi_{m+1}^{v_{m+1}}(v_1, \dots, v_{m+1}), \quad H \models \neg(\varphi_{m+1}^{v_{m+1}}(u_1, \dots, u_{m+1})).$$

As in the previous case, there are a positive constant C (which does not depend on $|V(G)|$, $|V(H)|$) and a set $\mathcal{V} \subset V(H)$ with $|\mathcal{V}| \leq C$ such that the following property holds. For any $v_{m+1} \in V(G)$, there exists $v \in \mathcal{V}$ such that $\varphi_{m+1}^v(x_1, \dots, x_{m+1}) \cong \varphi_{m+1}^{v_{m+1}}(x_1, \dots, x_{m+1})$. Set

$$\varphi_m(x_1, \dots, x_m) = \forall x_{m+1} \left(\bigvee_{v \in \mathcal{V}} (\varphi_{m+1}^v(x_1, \dots, x_{m+1})) \right).$$

By the induction, we get that $\phi = \phi_0$ is the required sentence which is true for G and false for H . Obviously, $\text{ch}(\phi) \leq k$. \square

Lemma 6 *The following two properties are equivalent:*

- 1) Spoiler has a winning strategy in $\text{EHR}(G, H, q, k)$;
- 2) there is $\phi \in \mathcal{F}$ with $q(\phi) = q$ such that a number of labels alternations in any path of $F(\phi)$ on q vertices starting in a root equals k , and $G \models \phi$, $H \models \neg(\phi)$.

Proof. First, let us prove that 2) implies 1). The winning strategy of Spoiler is absolutely the same as in the proof of Lemma 5. The only thing we should prove is that Spoiler alternates exactly k times. If $\ell(q) < q$, then consider a path $\beta_1 \dots \beta_{\ell(q)} \beta_{\ell(q)+1} \dots \beta_q$ in $F(\phi)$. The number of labels alternations in this path equals k . Therefore, $k - \tilde{k} \leq q - \ell(q)$. So, Spoiler can choose graphs (and an arbitrary vertex) in each of the remaining rounds in a way such that he will alternate k times overall. If $\ell(q) = q$, then, obviously, $\tilde{k} = k$.

It remains to prove that 1) implies 2). The formula ϕ is constructed in the same way as in the proof of Lemma 5. We only need to prove that $\text{ch}(\phi) = k$. Consider an arbitrary path $\beta_1 \dots \beta_q$ in $F(\phi)$ starting in a root. Note that β_i is labeled by \exists if and only if there exists a Duplicator's strategy such that in the i -th round Spoiler chooses G . Therefore, the number of labels alternations in this path equals k . \square

4 Spectra of formulas with small numbers of alternations

Let us start this section with the following simple observation.

Lemma 7 *If $\phi \in \mathcal{F}$ and $\alpha \in S(\phi)$, then there exists an NEPNF sentence $\hat{\phi}$ such that $\text{ch}(\phi) = \text{ch}(\hat{\phi})$ and $\alpha \in S(\hat{\phi})$ as well.*

Proof. By Lemma 3, it is enough to prove that there exists a sentence $\hat{\phi} = \exists x (\varphi(x)) \in \mathcal{F}$ such that α belongs to its spectrum.

As $\alpha \in S(\phi)$, there exist $\varepsilon > 0$ and sequences n_i, m_i such that, for any $i \in \mathbb{N}$,

$$\min \left\{ \mathbf{P} \left(G(n_i, n_i^{-\alpha}) \models \phi \right), \mathbf{P} \left(G(m_i, m_i^{-\alpha}) \models \neg(\phi) \right) \right\} > \varepsilon.$$

Fix $i \in \mathbb{N}$. Let G, H be graphs on n_i, m_i vertices respectively such that $G \models \phi$, $H \models \neg(\phi)$. The formula ϕ is a logical combination (disjunctions and conjunctions) of formulas

$$\exists x (\varphi_j(x)), \quad \forall x (\varphi^j(x)).$$

Let N be the number of all formulas in this combination. Obviously, there exists either j such that $G \models \exists x (\varphi_j(x))$, $H \models \neg(\exists x (\varphi_j(x)))$ or j such that $G \models \forall x (\varphi^j(x))$, $H \models \neg(\forall x (\varphi^j(x)))$. Therefore, there exists $\varphi(x) = \varphi(x, i) \in \mathcal{F}$ such that $\text{ch}(\exists x (\varphi(x))) \leq \text{ch}(\phi)$, $\text{q}(\exists x (\varphi(x))) \leq \text{q}(\phi)$ and

$$\min \left\{ \mathbf{P} \left(G(n_i, n_i^{-\alpha}) \models \exists x (\varphi(x)) \right), \mathbf{P} \left(G(m_i, m_i^{-\alpha}) \models \neg(\exists x (\varphi(x))) \right) \right\} > \varepsilon/N.$$

Set $\hat{\phi}_i = \exists x (\varphi(x, i))$. As there is only a finite number of representatives of \cong -equivalence classes of sentences in \mathcal{F} with a fixed quantifier depth (see, e.g., [16]), there is only a finite number of representatives of \cong -equivalence classes in $\{\hat{\phi}_i, i \in \mathbb{N}\}$ as well. Therefore, there exists a sentence $\hat{\phi} = \exists x (\varphi(x))$ and a sequence i_j such that, for all $j \in \mathbb{N}$,

$$\min \left\{ \mathbf{P} \left(G(n_{i_j}, n_{i_j}^{-\alpha}) \models \hat{\phi} \right), \mathbf{P} \left(G(m_{i_j}, m_{i_j}^{-\alpha}) \models \neg(\hat{\phi}) \right) \right\} > \varepsilon/N.$$

So, $\alpha \in S(\hat{\phi})$. \square

Below, we state the main result of this section, which implies the following answer on Q2:

the minimal number of quantifier alternations of a first order sentence with an infinite spectrum equals 3.

Theorem 4 *The minimal k such that there exists $\phi \in \mathcal{F}$ with infinite $S(\phi)$ and $\text{ch}(\phi) = k$ equals 3.*

Proof. By Lemma 7 and Theorem 1, it is enough to prove that, for any $k \in \{0, 1, 2\}$ and any NEPNF sentence $\phi = \exists x (\varphi(x)) \in \mathcal{F}$ with $\text{ch}(\phi) = k$, the set $S(\phi)$ is finite. Note that $S(\phi) = S(\neg(\phi))$. Therefore, equivalently, we may prove that spectra of sentences $\forall x (\varphi(x))$ are finite.

Obviously, $k \in \{0, 1\}$ are subcases of $k = 2$. However, below we consider $k = 0$, $k = 1$ alone for the sake of convenience.

Let $\phi_H \in \mathcal{F}$ be an existence sentence which expresses the property of containing an induced subgraph isomorphic to H .

4.1 No alternations

Let $\text{ch}(\phi) = 0$, where $\phi = \exists x (\varphi(x)) \in \mathcal{F}$ is an NEPNF sentence. Obviously, there exists a finite set \mathcal{G} of graphs such that $G \models \phi$ if and only if in G there is an induced subgraph which is isomorphic to some $H \in \mathcal{G}$. We get

$$\phi \cong \bigvee_{H \in \mathcal{G}} (\phi_H).$$

By Theorem 2, either $\rho := \min_{H \in \mathcal{G}} \{\rho(H)\} > 0$ and $S(\phi) \subset \{1/\rho\}$, or $\rho = 0$ and $S(\phi) = \emptyset$.

4.2 One alternation

Let $\text{ch}(\phi) = 1$, where

$$\phi = \forall y_1 \dots \forall y_s \exists x_1 \dots \exists x_m (\varphi(y_1, \dots, y_s, x_1, \dots, x_m)) \in \mathcal{F}$$

has the quantifier depth $s + m$. Obviously, there exists a finite set \mathcal{G} of graphs on a set of vertices $\{a_1, \dots, a_s\}$ and, for each $A \in \mathcal{G}$, there exists a finite set $\mathcal{H}(A)$ of graphs on a set of vertices $\{a_1, \dots, a_s, b_1, \dots, b_m\}$ such that

- for any $A \in \mathcal{G}$ and $B \in \mathcal{H}(A)$, $A = B|_{\{a_1, \dots, a_s\}}$,
- $G \models \phi$ if and only if, for any distinct vertices $y_1, \dots, y_s \in V(G)$, there exist distinct vertices $x_1, \dots, x_m \in V(G)$ ($x_j \neq y_i$) and graphs $A \in \mathcal{G}$, $B \in \mathcal{H}(A)$ such that the map $f : B \rightarrow G|_{\{y_1, \dots, y_s, x_1, \dots, x_m\}}$, $f(a_i) = y_i$, $f(b_j) = x_j$, is an isomorphism.

Let all graphs A_1, \dots, A_M of \mathcal{G} be ordered in a way such that

$$\rho_1 := \rho(A_1) \geq \dots \geq \rho(A_M) =: \rho_M. \quad (2)$$

For each $i \in \{1, \dots, M\}$, let $\rho^i = \min\{\rho(B, A_i), B \in \mathcal{H}(A_i)\}$.

Suppose that $1/\alpha$ is not equal to any of $\rho_i, \rho^i, i \in \{1, \dots, M\}$. If there is a graph on the set of vertices $\{a_1, \dots, a_s\}$ which does not belong to \mathcal{G} such that its maximal density is less than $1/\alpha$, then, by Theorem 2, $G(n, p) \models \neg(\phi)$ (a.a.s.). Suppose that the above property does not hold. This implies that $\rho_M = 0$. Set $\rho_0 = \infty, 1/\rho_0 = 0$ and $1/\rho_M = \infty$. Let $i_0 \in \{0, 1, \dots, M-1\}$ be chosen in the following way: $1/\rho_{i_0} < \alpha < 1/\rho_{i_0+1}$. If for some $i \in \{i_0 + 1, \dots, M\}$ the inequality $\rho^i > 1/\alpha$ holds, then, by Theorem 2, $G(n, p) \models \neg(\phi)$ (a.a.s.). Otherwise, $G(n, p) \models \phi$ (a.a.s.). Thus, $S(\phi) \subseteq \{1/\rho_1, \dots, 1/\rho_M, 1/\rho^1, \dots, 1/\rho^M\}$, and so $|S(\phi)| < \infty$.

4.3 Two alternations

In this case, it is not enough to define sets of graphs as above. We divide the proof into four parts. Only the first part “Transition to sets of graphs” is similar to the previous cases.

4.3.1 Transition to sets of graphs

Let a sentence

$$\phi = \forall y_1 \dots \forall y_s \exists x_1 \dots \exists x_m \forall w_1 \dots \forall w_r (\varphi(y_1, \dots, y_s, x_1, \dots, x_m, w_1, \dots, w_r)) \in \mathcal{F}$$

has the quantifier depth $s + m + r$.

Obviously, there exists a set of vertices $\Sigma = \Sigma_a \sqcup \Sigma_b \sqcup \Sigma_c$, where $\Sigma_a = \{a_1, \dots, a_s\}$, $\Sigma_b = \{b_1, \dots, b_m\}$, $\Sigma_c = \{c_1, \dots, c_r\}$, and

- a finite set of graphs \mathcal{G} on the set of vertices Σ_a ,
- for each $A \in \mathcal{G}$, a finite set $\mathcal{H}(A)$ on the set of vertices $\Sigma_a \sqcup \Sigma_b$,
- for each $A \in \mathcal{G}$ and $B \in \mathcal{H}(A)$, a finite set of graphs $\mathcal{K}(B)$ on the set of vertices Σ ,

such that the following properties hold.

- For any $A \in \mathcal{G}$, $B \in \mathcal{H}(A)$, $C \in \mathcal{K}(B)$, we have $A = B|_{\Sigma_a}$, $B = C|_{\Sigma_a \sqcup \Sigma_b}$.
- $G \models \phi$ if and only if for any pairwise distinct y_1, \dots, y_s from $V(G)$ there exist pairwise distinct x_1, \dots, x_m from $V(G) \setminus \{y_1, \dots, y_s\}$ such that for any pairwise distinct w_1, \dots, w_r from $V(G) \setminus \{y_1, \dots, y_s, x_1, \dots, x_m\}$ the graph G has the property $P(y_1, \dots, y_s, x_1, \dots, x_m, w_1, \dots, w_r)$ (which is defined below).

Let us say that G has the property $P(y_1, \dots, y_s, x_1, \dots, x_m, w_1, \dots, w_r)$, if there exist graphs $A \in \mathcal{G}$, $B \in \mathcal{H}(A)$, $C \in \mathcal{K}(B)$ such that the map $f : C \rightarrow G|_{\{y_1, \dots, y_s, x_1, \dots, x_m, w_1, \dots, w_r\}}$ which preserves the orders of the vertices ($f(a_i) = y_i$, $f(b_j) = x_j$, $f(c_h) = w_h$) is an isomorphism.

Theorem 3 from [7] implies that $\alpha \notin S(\phi)$ for any $\alpha < \frac{1}{s+m+r-2}$. Therefore, for any positive integer N , the set of numbers from $S(\phi)$ with a numerator at most N is finite. So, we may assume that the numerator of α is large enough. As in the case of one alternation, we assume that any graph on the set of vertices Σ_a with a maximal density less than $1/\alpha$ belongs to \mathcal{G} .

4.3.2 Dense neighbourhood and its structure

Let Γ be an arbitrary graph on a set of vertices V with the following property. There is $A \in \mathcal{G}$ and pairwise distinct vertices $y_1, \dots, y_s \in V$ such that the map $A \rightarrow \Gamma|_{\{y_1, \dots, y_s\}}$ which preserves the orders of the vertices is an isomorphism.

Let $Y_0 = \Gamma|_{\{y_1, \dots, y_s\}}$. For each $i \geq 0$, let us construct an induced subgraph Y_{i+1} of Γ on the union of $V(Y_i)$ with some additional vertices (for a step \tilde{i} , this process halts, set $Y = Y_{\tilde{i}}$). For a step i the process *does not halt*, if there exists a subgraph $W \subset \Gamma$ such that $W \supset Y_i$, $v(W) - v(Y_i) \leq r$ and $\rho(W, Y_i) > 1/\alpha$. For such a graph W , set $Y_{i+1} = W$.

The graph $Y = Y(\Gamma; y_1, \dots, y_s)$ is constructed. Before proceeding with the next part of the proof, let us study a structure of Y and introduce some notations for describing this structure.

- Let $\mathcal{U} = \mathcal{U}(A) = \{B_1, \dots, B_\beta\}$ be the set of all graphs B on the set of vertices $\Sigma_a \cup \Sigma_b$ such that $B|_{\Sigma_a} = A$. Obviously, $\beta = 2^{C_m^2 + sm}$.
- Let x_1^0, \dots, x_m^0 be arbitrary vertices which are not in $V(Y)$ (and even not necessarily in V).
- Let $\ell \in \{1, \dots, \beta\}$, $\tilde{m} \in \{0, \dots, m\}$. Consider the set $\mathcal{X}_{\ell, \tilde{m}}$ of all collections of vertices $x_1, \dots, x_{\tilde{m}} \in V(Y) \setminus \{y_1, \dots, y_s\}$ such that there exists a graph W on the set of vertices $V(Y) \cup \{x_{\tilde{m}+1}^0, \dots, x_m^0\}$ and an isomorphism $f : B_\ell \rightarrow W|_{\{y_1, \dots, y_s, x_1, \dots, x_{\tilde{m}}, x_{\tilde{m}+1}^0, \dots, x_m^0\}}$ which preserves the orders of the vertices ($f(a_i) = y_i$, $f(b_j) \in \{x_j, x_j^0\}$).
- For each $\ell \in \{1, \dots, \beta\}$, $\tilde{m} \in \{0, \dots, m\}$, $(x_1, \dots, x_{\tilde{m}}) \in \mathcal{X}_{\ell, \tilde{m}}$, consider the set $\mathcal{S}_\ell(x_1, \dots, x_{\tilde{m}})$ of all graphs W on the set of vertices $V(Y) \cup \{x_{\tilde{m}+1}^0, \dots, x_m^0\}$ such that $W|_{V(Y)} = Y$, $\rho(W, Y) < 1/\alpha$ and the map $f : B_\ell \rightarrow W|_{\{y_1, \dots, y_s, x_1, \dots, x_{\tilde{m}}, x_{\tilde{m}+1}^0, \dots, x_m^0\}}$

which preserves the orders of the vertices ($f(a_i) = y_i$, $f(b_j) \in \{x_j, x_j^0\}$) is an isomorphism. Moreover, for each $W \in \mathcal{S}_\ell(x_1, \dots, x_{\tilde{m}})$ consider the set $\mathcal{N}_\ell(W; x_1, \dots, x_{\tilde{m}})$ of all graphs C on the sets of vertices $\Sigma_a \cup \Sigma_b \cup \{c_1, \dots, c_{\tilde{r}}\}$, where $\tilde{r} \leq r$, such that there exists a subgraph $Z \subset W$ containing the vertices $y_1, \dots, y_s, x_1, \dots, x_{\tilde{m}}, x_{\tilde{m}+1}^0, \dots, x_m^0$, and the following two properties hold. First, there exist vertices $w_1, \dots, w_{\tilde{r}} \in V(Y)$ and an isomorphism $f : C \rightarrow Z$ which preserves the orders of the vertices ($f(a_i) = y_i$, $f(b_j) \in \{x_j, x_j^0\}$, $f(c_h) = w_h$). Second,

$$\rho\left(Z, Z|_{\{y_1, \dots, y_s, x_1, \dots, x_{\tilde{m}}, x_{\tilde{m}+1}^0, \dots, x_m^0\}}\right) > 1/\alpha.$$

- For each $\ell \in \{1, \dots, \beta\}$, denote by $(\mathcal{N})_\ell[Y; y_1, \dots, y_s]$ a maximal set of pairwise distinct sets among $\mathcal{N}_\ell(W; x_1, \dots, x_{\tilde{m}})$, $W \in \mathcal{S}_\ell$.

The vector $\mathbf{N} = ((\mathcal{N})_1[Y; y_1, \dots, y_s], \dots, (\mathcal{N})_\beta[Y; y_1, \dots, y_s])$ defines the structure of Y .

4.3.3 Existence of a bounded graph with the same structure

Let $\{y_1, \dots, y_s\}$ be an arbitrary set of vertices, and $A \in \mathcal{G}$.

Consider an arbitrary graph Γ which contains the vertices y_1, \dots, y_s such that the map $A \rightarrow \Gamma|_{\{y_1, \dots, y_s\}}$ (preserving the orders of the vertices) is an isomorphism. Let $\ell \in \{1, \dots, \beta\}$ (where β is the cardinality of $\mathcal{U}(A) = \{B_1, \dots, B_\beta\}$). Determine the vector $(\mathcal{N})_\ell := (\mathcal{N})_\ell[Y(\Gamma; y_1, \dots, y_s); y_1, \dots, y_s]$. Let $\mathbf{Y} = \mathbf{Y}(\Gamma; y_1, \dots, y_s)$ be the set of all graphs Y such that $Y|_{\{y_1, \dots, y_s\}} = \Gamma|_{\{y_1, \dots, y_s\}}$, and $(\mathcal{N})_\ell = (\mathcal{N})_\ell[Y; y_1, \dots, y_s]$ for all $\ell \in \{1, \dots, \beta\}$. Let the graph $Y_{\min}(\mathbf{Y}; y_1, \dots, y_s)$ has a minimal number of vertices among the graphs in the set

$$\{Y \in \mathbf{Y} : \forall \tilde{Y} \in \mathbf{Y} (\rho(\tilde{Y}) \geq \rho(Y))\}$$

(and, of course, belongs to this set).

Note that the set $\mathbf{Y}(\Gamma; y_1, \dots, y_s)$ is defined by the vector $\mathbf{N} = ((\mathcal{N})_1, \dots, (\mathcal{N})_\beta)$ only. Therefore, for the vertices y_1, \dots, y_s there exist only finite set of pairwise distinct sets $\mathbf{Y}(\cdot; y_1, \dots, y_s)$. So, the set of pairwise distinct graphs $Y_{\min}(\cdot; y_1, \dots, y_s)$ is finite. Let

$$Y_{\min}^1(y_1, \dots, y_s), \dots, Y_{\min}^\theta(y_1, \dots, y_s)$$

be all such graphs.

4.3.4 Finiteness of the spectrum

Recall that the numerator of the irreducible fraction $\alpha = \frac{R}{P}$ is large enough (see Section 4.3.1). So, we assume that $R > \max\{s + m + r, N\}$, where

$$N := \max \left\{ v(Y_{\min}^1(y_1, \dots, y_s)), \dots, v(Y_{\min}^\theta(y_1, \dots, y_s)) \right\}.$$

Note that N does not depend on a choice of y_1, \dots, y_s .

Theorems 2, 3 imply that a.a.s. the random graph $G(n, n^{-\alpha})$ has the following properties:

- G1 for any H with $\rho(H) > 1/\alpha$ and $v(H) \leq s + r(sP + 1)$, there is no subgraph isomorphic to H ;
- G2 for any $H \subset G$ with $v(G) \leq s + m + r$ and $\rho(G, H) < 1/\alpha$, there is the (G, H) -extension property;
- G3 for any $H \subset G$ with $v(G) \leq \max\{N, m + s + rsP\}$, $\rho(G, H) < 1/\alpha$ and set \mathcal{W} of graphs W on a fixed set of vertices such that
 - $G \subset W$, $1 \leq v(W) - v(G) \leq r + m$, $\rho(W, G) > 1/\alpha$,
 - $W \setminus G$ is connected,
 - there are edges between $W \setminus G$ and G in W ,

there is the (\mathcal{W}, G, H) -double-extension property.

Let us prove that if the graphs Γ, Υ have the properties G1, G2 and G3, then either ϕ is true for both of them, or ϕ is false for both of them. This would imply that $\alpha \notin S(\phi)$.

Assume that $\Gamma \models \neg(\phi)$, $\Upsilon \models \phi$. By the property G1, a maximal density of any subgraph of Γ on s vertices is less than $1/\alpha$. All graphs on the set of vertices Σ_a with such a maximal density are in \mathcal{G} (see Section 4.3.1). Therefore, there exist $A \in \mathcal{G}$ and pairwise distinct $y_1, \dots, y_s \in V(\Gamma)$ such that the map $A \rightarrow \Gamma|_{\{y_1, \dots, y_s\}}$ which preserves the orders of the vertices is an isomorphism, and Γ with distinguished vertices y_1, \dots, y_s *does not have* the property (EXT), which is defined below.

(EXT): *there exist pairwise distinct $x_1, \dots, x_m \in V(\Gamma) \setminus \{y_1, \dots, y_s\}$ such that for any pairwise distinct $w_1, \dots, w_r \in V(\Gamma) \setminus \{y_1, \dots, y_s, x_1, \dots, x_m\}$ there exist graphs $B \in \mathcal{H}(A)$, $C \in \mathcal{K}(B)$ and an isomorphism $f : C \rightarrow \Gamma|_{\{y_1, \dots, y_s, x_1, \dots, x_m, w_1, \dots, w_r\}}$ which preserves the orders of the vertices ($f(a_i) = y_i$, $f(b_j) = x_j$, $f(c_h) = w_h$).*

Construct the graph $Y = Y(\Gamma; y_1, \dots, y_s)$ as it is done in Section 4.3.2. Let us prove that $v(Y) \leq s + rsP$. Assume that the opposite inequality is true. By the definition of Y , there is a subgraph $X \subset Y$ on at most $s + r(sP + 1)$ vertices such that, for some $v_1, \dots, v_{sP+1} \in \{1, \dots, r\}$,

$$\rho(X) \geq \frac{(1/\alpha)v_1 + \dots + (1/\alpha)v_{sP+1} + \frac{sP+1}{R}}{s + v_1 + \dots + v_{sP+1}} = \frac{1}{\alpha} + \frac{1}{R(s + v_1 + \dots + v_{sP+1})} > \frac{1}{\alpha}.$$

This contradicts the property G1. So, $v(Y) \leq s + rsP$, and, therefore, $\rho(Y) \leq 1/\alpha$.

Consider the graph $Y_{\min} = Y_{\min}(\mathbf{Y}(\Gamma; y_1, \dots, y_s); y_1, \dots, y_s)$. We have $\rho(Y_{\min}) \leq \rho(Y) \leq 1/\alpha$. As $v(Y_{\min}) \leq N < R$, the equality $\rho(Y_{\min}) = 1/\alpha$ is impossible, and so $\rho(Y_{\min}) < 1/\alpha$.

By the property G3, in Υ there is an induced subgraph $Y^\Upsilon \cong Y_{\min}$ such that in Υ there is no subgraph $W \supset Y^\Upsilon$ with $v(W) - v(Y^\Upsilon) \leq r + m$ and $\rho(W, Y^\Upsilon) > 1/\alpha$. Let $f : Y_{\min} \rightarrow Y^\Upsilon$ be an isomorphism. Set $f(y_i) = y_i^\Upsilon$, $i \in \{1, \dots, s\}$. As $\Upsilon \models \phi$, Υ with distinguished vertices $y_1^\Upsilon, \dots, y_s^\Upsilon$ has the property (EXT). Let $x_1^\Upsilon, \dots, x_m^\Upsilon \in V(\Upsilon) \setminus \{y_1^\Upsilon, \dots, y_s^\Upsilon\}$ and $B \in \mathcal{H}(A)$ be such that for any pairwise distinct $w_1^\Upsilon, \dots, w_r^\Upsilon \in V(\Upsilon) \setminus \{y_1^\Upsilon, \dots, y_s^\Upsilon, x_1^\Upsilon, \dots, x_m^\Upsilon\}$ there exist a graph $C \in \mathcal{K}(B)$ and an isomorphism $g : C \rightarrow \Upsilon|_{\{y_1^\Upsilon, \dots, y_s^\Upsilon, x_1^\Upsilon, \dots, x_m^\Upsilon, w_1^\Upsilon, \dots, w_r^\Upsilon\}}$ which preserves the orders of the vertices ($g(a_i) = y_i^\Upsilon$, $g(b_j) = x_j^\Upsilon$, $g(c_h) = w_h^\Upsilon$).

From the property G1 it follows that

$$\rho\left(\Upsilon|_{V(Y^\Upsilon) \cup \{x_1^\Upsilon, \dots, x_m^\Upsilon\}}, Y^\Upsilon\right) < 1/\alpha$$

if at least one of the vertices $x_1^\Upsilon, \dots, x_m^\Upsilon$ is not in Y^Υ . Indeed, there is no equality, because $v\left(\Upsilon|_{V(Y^\Upsilon) \cup \{x_1^\Upsilon, \dots, x_m^\Upsilon\}}\right) - v(Y^\Upsilon) \leq m$. Let $x_1, \dots, x_{\tilde{m}} \in Y^\Upsilon$, $x_{\tilde{m}+1}, \dots, x_m \in V(\Upsilon) \setminus V(Y^\Upsilon)$, where $\tilde{m} \in \{0, 1, \dots, m\}$. From the property G3, the definitions of Y and Y_{\min} it follows that there exist vertices $x_1, \dots, x_{\tilde{m}} \in V(Y)$, $x_{\tilde{m}+1}, \dots, x_m \in V(\Gamma) \setminus V(Y)$ such that the following properties hold.

Q1 There exists an isomorphism $f : B \rightarrow \Gamma|_{\{y_1, \dots, y_s, x_1, \dots, x_m\}}$ which preserves the orders of the vertices ($f(a_i) = y_i$, $f(b_j) = x_j$).

Q2 There is no $W \subset \Gamma$ such that $W \supset \Gamma|_{V(Y) \cup \{x_1, \dots, x_m\}}$,

$$v(W) - v(\Gamma|_{V(Y) \cup \{x_1, \dots, x_m\}}) \leq r \quad \text{and} \quad \rho(W, \Gamma|_{V(Y) \cup \{x_1, \dots, x_m\}}) > 1/\alpha.$$

Q3 Let C be a graph on a set of vertices $\{a_1, \dots, a_s, b_1, \dots, b_m, c_1, \dots, c_{\tilde{r}}\}$ (where $\tilde{r} \leq r$). Let $Z \subseteq \Gamma$ be a graph consisting of the vertices $y_1, \dots, y_s, x_1, \dots, x_m$ and some vertices

$w_1, \dots, w_{\tilde{r}} \in V(Y)$. Moreover, let the map $f : C \rightarrow Z$ which preserves the orders of the vertices ($f(a_i) = y_i$, $f(b_j) = x_j$, $f(c_h) = w_h$) be an isomorphism, and

$$\rho(Z, Z|_{\{y_1, \dots, y_s, x_1, \dots, x_m\}}) > 1/\alpha.$$

Then, in Υ there is a subgraph Z^Υ consisting of the vertices $y_1^\Upsilon, \dots, y_s^\Upsilon, x_1^\Upsilon, \dots, x_m^\Upsilon$ and some vertices $w_1^\Upsilon, \dots, w_{\tilde{r}}^\Upsilon \in V(Y^\Upsilon)$ such that the map $f : C \rightarrow Z^\Upsilon$ which preserves the orders of the vertices ($f(a_i) = y_i^\Upsilon$, $f(b_j) = x_j^\Upsilon$, $f(c_h) = w_h^\Upsilon$) is an isomorphism.

By our assumption, there exist $w_1, \dots, w_r \in V(\Gamma) \setminus \{y_1, \dots, y_s, x_1, \dots, x_m\}$ such that for any $C \in \mathcal{K}(B)$ the map $f : C \rightarrow \Gamma|_{\{y_1, \dots, y_s, x_1, \dots, x_m, w_1, \dots, w_r\}}$ which preserves the orders of the vertices ($f(a_i) = y_i$, $f(b_j) = x_j$, $f(c_h) = w_h$) is not an isomorphism. If

$$\rho(\Gamma|_{\{y_1, \dots, y_s, x_1, \dots, x_m, w_1, \dots, w_r\}}, \Gamma|_{\{y_1, \dots, y_s, x_1, \dots, x_m\}}) < 1/\alpha, \quad (3)$$

then by the property G2 in Υ there are vertices $w_1^\Upsilon, \dots, w_r^\Upsilon$ such the the map

$$f : \Gamma|_{\{y_1, \dots, y_s, x_1, \dots, x_m, w_1, \dots, w_r\}} \rightarrow \Upsilon|_{\{y_1^\Upsilon, \dots, y_s^\Upsilon, x_1^\Upsilon, \dots, x_m^\Upsilon, w_1^\Upsilon, \dots, w_r^\Upsilon\}} \quad (4)$$

which preserves the orders of the vertices ($f(y_i) = y_i^\Upsilon$, $f(x_j) = x_j^\Upsilon$, $f(w_h) = w_h^\Upsilon$) is an isomorphism — a contradiction.

If $w_1, \dots, w_r \in V(\Gamma) \setminus V(Y)$, then Inequality (3) holds (there is no equality, because $v(\Gamma|_{\{y_1, \dots, y_s, x_1, \dots, x_m, w_1, \dots, w_r\}}) - v(\Gamma|_{\{y_1, \dots, y_s, x_1, \dots, x_m\}}) = r < R$).

If $w_1, \dots, w_r \in V(Y)$ and

$$\rho(\Gamma|_{\{y_1, \dots, y_s, x_1, \dots, x_m, w_1, \dots, w_r\}}, \Gamma|_{\{y_1, \dots, y_s, x_1, \dots, x_m\}}) > 1/\alpha,$$

then, the definition of Y^Υ implies the existence of vertices $w_1^\Upsilon, \dots, w_r^\Upsilon$ such that the map (4) which preserves the orders of the vertices is an isomorphism — a contradiction.

Finally, let some (not all) of the vertices w_1, \dots, w_r be in $V(Y)$ (say, $w_1, \dots, w_{\tilde{r}} \in V(Y)$, $w_{\tilde{r}+1}, \dots, w_r \in V(\Gamma) \setminus V(Y)$). In Y^Υ there exist vertices $w_1^\Upsilon, \dots, w_{\tilde{r}}^\Upsilon$ such that the map $f : \Gamma|_{\{y_1, \dots, y_s, x_1, \dots, x_m, w_1, \dots, w_{\tilde{r}}\}} \rightarrow \Upsilon|_{\{y_1^\Upsilon, \dots, y_s^\Upsilon, x_1^\Upsilon, \dots, x_m^\Upsilon, w_1^\Upsilon, \dots, w_{\tilde{r}}^\Upsilon\}}$ which preserves the orders of the vertices is an isomorphism. Moreover,

$$\rho(\Gamma|_{\{y_1, \dots, y_s, x_1, \dots, x_m, w_1, \dots, w_r\}}, \Gamma|_{\{y_1, \dots, y_s, x_1, \dots, x_m, w_1, \dots, w_{\tilde{r}}\}}) < 1/\alpha.$$

By the property G2, in Υ there exist vertices $w_{\tilde{r}+1}^\Upsilon, \dots, w_r^\Upsilon$ such that the map (4) which preserves the orders of the vertices is an isomorphism — a contradiction. \square

5 Spectra of formulas with small quantifier depths

Theorem 4 answers the second question of Section 1. We do not have an answer on the third question. However, in our second main result, we get a new lower bound on the minimal quantifier depth of PNF sentence with an infinite spectrum.

Theorem 5 *The minimal q such that there exists a PNF sentence $\phi \in \mathcal{F}$ with infinite $S(\phi)$ and $q(\phi) = q$ is at least 5.*

The proof is based on the statement on Ehrenfeucht game which is given below. For a positive integer k , consider a set $\tilde{S}(k)$ of $\alpha > 0$ such that there exist $\varepsilon > 0$ and increasing sequences n_i, m_i of positive integers such that, for any $i \in \mathbb{N}$,

$$P \left(\text{Spoiler has a winning strategy in EHR} \left(G(n_i, n_i^{-\alpha}), G(m_i, m_i^{-\alpha}), k, k-1 \right) \right) > \varepsilon^2.$$

Lemma 8 *The set $\tilde{S}(4) \cap (1/2, 1)$ is finite.*

Proof. Case 1. Let $p = n^{-\alpha}$, $\alpha \in (1/2, 10/19)$.

Let x_1, x_2, x_3 be vertices of an arbitrary graph G . For any $i, j \in \{\{0\}, \mathbb{N}\}$, we say that (x_1, x_2, x_3) has the type (i, j) , if a number of common neighbors of x_1, x_3 (which are not adjacent to x_2) is in i , and a number of common neighbors of x_2, x_3 (which are not adjacent to x_1) is in j . Introduce a linear order \leq on the set \mathcal{I} of all pairs of elements from $\{\{0\}, \mathbb{N}\}$ in the following way: $(\{0\}, \{0\}) \leq (\{0\}, \mathbb{N}) \leq (\mathbb{N}, \{0\}) \leq (\mathbb{N}, \mathbb{N})$.

For any vertices x_1, x_2 , denote by $n(x_1, x_2)$ the number of all pairs of adjacent common neighbors of x_1, x_2 . Denote the set of all common neighbors of x_1, x_2 by $N(x_1, x_2)$. Denote the set of all common neighbors x_3 of x_1, x_2 such that x_1, x_2, x_3 have no common neighbors by $U(x_1, x_2)$.

We say that a graph has the *triangle property*, if, for any $s \in \{0, 1, 2\}$, any vertex x_1 , any $x, y \in \mathcal{I}$ and any $\delta \in \{\sim, \approx\}$, there is a vertex x_2 in the graph such that

- $x_1 \delta x_2$,
- $n(x_1, x_2) \leq 1$,
- there is no K_4 containing x_1, x_2 ,
- if $n(x_1, x_2) = 1$, then $|U(x_1, x_2)| = \min\{s, 1\}$,
- if $n(x_1, x_2) = 0$, then $|U(x_1, x_2)| = s$,

- for any $x_3 \in U(x_1, x_2)$, (x_1, x_2, x_3) has the type x ,
- for any $x_3 \in N(x_1, x_2) \setminus U(x_1, x_2)$, (x_1, x_2, x_3) has the type y .

By Theorem 3, a.a.s. $G(n, p)$ has the triangle property. Moreover, by Theorem 3, a.a.s. $G(n, p)$ has the *sparse extension property*, which is described below. For any $m \geq 1$ and any distinct vertices v_1, \dots, v_m , there are vertices z_1, z_2 such that

- z_1 is adjacent to v_1 and not adjacent to any of v_2, \dots, v_m , z_2 is not adjacent to any of v_1, \dots, v_m ,
- for any $i \in \{1, \dots, m\}$, $s \in \{1, 2\}$, $z_s \neq v_i$ and z_s has no common neighbors with v_i .

Finally, by Theorem 3, a.a.s., in $G(n, p)$, there exists a vertex x_1 such that

- there is no K_4 containing x_1 ,
- for any vertex x_2 , $n(x_1, x_2) \leq 1$

(in such a case, we say that a graph has the *sparse subgraph property*).

Let G, H be graphs with the triangle property, the sparse extension property and the sparse subgraph property. Let us describe a winning strategy of Duplicator in $\text{EHR}(G, H, 4, 3)$.

In the first round, Spoiler chooses, say, an arbitrary vertex $v_1 \in V(G)$. Duplicator chooses an arbitrary vertex $u_1 \in V(H)$ such that there is no K_4 in H containing u_1 and, for any vertex u_2 , $n(u_1, u_2) \leq 1$. Such a vertex exists because H has the sparse subgraph property.

In the second round, Spoiler chooses a vertex $u_2 \in V(H)$. If the set $N(u_1, u_2) \setminus U(u_1, u_2)$ is non-empty, then denote by $y \in \mathcal{I}$ the least element of the set of types of (u_1, u_2, u_3) over all $u_3 \in N(u_1, u_2) \setminus U(u_1, u_2)$. If the set $U(u_1, u_2)$ is non-empty, then denote by $x \in \mathcal{I}$ the least element of the set of types of (u_1, u_2, u_3) over all $u_3 \in U(u_1, u_2)$.

Consider two cases.

1. $u_1 \sim u_2$. Duplicator chooses $v_2 \in V(G)$ such that

- $v_1 \sim v_2$,
- there is no K_4 containing v_1, v_2 in G ,
- for any $s \in \{0, 1\}$, v_1, v_2 have exactly s common neighbors if and only if u_1, u_2 have exactly s common neighbors,
- if u_1, u_2 have 2 common neighbors, then v_1, v_2 have exactly 2 common neighbors,

- if $N(u_1, u_2) \neq \emptyset$, then the types of (v_1, v_2, v_3) equal x for all common neighbors v_3 of v_1, v_2 .

2. $u_1 \approx u_2$. Duplicator chooses $v_2 \in V(G)$ such that

- $v_1 \approx v_2$,
- $n(v_1, v_2) = n(u_1, u_2)$,
- if $n(u_1, u_2) = 1$, then the types of (v_1, v_2, v_3^1) , (v_1, v_2, v_3^2) equal y , where $v_3^1 \sim v_3^2$ are common neighbors of v_1, v_2 ,
- if $n(u_1, u_2) = 1$ and $U(u_1, u_2) \neq \emptyset$, then $U(v_1, v_2) = \{v_3\}$ and the type of (v_1, v_2, v_3) equals x ,
- if $n(u_1, u_2) = 0$ and $|U(u_1, u_2)| \geq 2$, then $U(v_1, v_2) = \{v_3^1, v_3^2\}$ and the types of (v_1, v_2, v_3^1) , (v_1, v_2, v_3^2) equal x ,
- if $n(u_1, u_2) = 0$ and $|U(u_1, u_2)| = 1$, then $U(v_1, v_2) = \{v_3\}$ and the type of (v_1, v_2, v_3) equals x ,
- if $N(u_1, u_2) = \emptyset$, then $N(v_1, v_2) = \emptyset$.

Such a vertex exists because 1) after the first round, there is no K_4 containing u_1 and $n(u_1, u_2) \leq 1$ for all u_2 ; 2) G has the triangle property.

In the third round, Spoiler chooses a vertex $v_3 \in V(G)$. If $v_3 \sim v_1, v_3 \sim v_2$, then Duplicator chooses a vertex $u_3 \in V(H)$ such that

- if $v_3 \in U(v_1, v_2)$, then $u_3 \in U(u_1, u_2)$ and (u_1, u_2, u_3) has the type x ,
- if $v_3 \in N(v_1, v_2) \setminus U(v_1, v_2)$, then $u_3 \in N(u_1, u_2) \setminus U(u_1, u_2)$ and (u_1, u_2, u_3) has the type y .

Otherwise, Duplicator chooses a vertex $u_3 \in V(H)$ such that

- $v_1 \sim v_3$ if and only if $u_1 \sim u_3$,
- $v_2 \sim v_3$ if and only if $u_2 \sim u_3$,
- for any $j \in \{1, 2\}$, the vertices u_j, u_3 have no common vertices.

Such a vertex exists because H has the sparse extension property.

In the fourth round, Spoiler chooses a vertex $u_4 \in V(H)$.

Obviously, if u_4 is a common neighbor of u_1, u_2, u_3 , then $u_1 \approx u_2$ and $u_3 \in N(u_1, u_2) \setminus U(u_1, u_2)$. Therefore, $v_3 \in N(v_1, v_2) \setminus U(v_1, v_2)$. So, there exists a common neighbor $v_4 \in V(G)$ of v_1, v_2, v_3 .

Assume that u_4 is not a common neighbor of u_1, u_2, u_3 . If $u_4 \in N(u_1, u_2)$, then v_1, v_2 have at least 1 common neighbor. So, if $u_3 \notin N(u_1, u_2)$, there is $v_4 \in N(v_1, v_2)$ such that $v_4 \neq v_3$. If $u_3 \in N(u_1, u_2)$, then v_1, v_2 have at least 2 common neighbors, and so there is $v_4 \in N(v_1, v_2)$ such that $v_4 \neq v_3$ as well. If $u_4 \in N(u_1, u_3)$ (or $u_4 \in N(u_2, u_3)$), then $u_3 \in N(u_1, u_2)$ and $(u_1, u_2, u_3), (v_1, v_2, v_3)$ has the same type. Therefore, there exists a vertex v_4 such that $v_4 \in N(v_1, v_3)$ (or $v_4 \in N(v_2, v_3)$).

In all the above cases, Duplicator chooses v_4 .

Finally, if u_4 is adjacent to at most one vertex of u_1, u_2, u_3 , then Duplicator has a winning strategy because G has the sparse extension property.

Case 2 Let $p = n^{-\alpha}$, $\alpha \in (10/19, 1)$ be rational and not equal to any fraction a/b with $a \leq 20$. Note that there is only a finite number of forbidden fractions a/b . Moreover, let q be the denominator of α .

Let $H_2 \subset H_1$, $V(H_2) = \{a_1, \dots, a_s\}$, $V(H_1) \setminus V(H_2) = \{b_1, \dots, b_\ell\}$. We say that a graph G has the *1-generic (H_1, H_2) -extension property* if for any its distinct vertices x_1, \dots, x_s there exist distinct vertices y_1, \dots, y_ℓ such that

- $\forall i, j \in \{1, \dots, \ell\} (y_i \sim y_j) \Leftrightarrow (b_i \sim b_j)$,
- $\forall i \in \{1, \dots, k\}, j \in \{1, \dots, \ell\} (x_i \sim y_j) \Leftrightarrow (a_i \sim b_j)$,
- if there exists a vertex z such that $\rho(G|_{\{x_1, \dots, x_s, y_1, \dots, y_\ell, z\}}, G|_{\{x_1, \dots, x_s, y_1, \dots, y_\ell\}}) > 1/\alpha$, then there are no edges between z and any of y_1, \dots, y_ℓ .

Let \mathcal{S} be a set of all graphs G , that satisfy the following properties.

- (1) There exists a vertex x in G such that there are no subgraphs $W \supset X$ with $x \in V(W)$, $\rho(W, (\{x\}, \emptyset)) > 1/\alpha$ and $v(W) \leq 21$.
- (2) For any graphs $H_2 \subset H_1$ with $\rho(H_1, H_2) < 1/\alpha$, $v(H_1) \leq 22$, G has the 1-generic (H_1, H_2) -extension property.

By Theorems 2 and 3, $\lim_{n \rightarrow \infty} \mathbf{P}(G(n, p) \in \mathcal{S}) = 1$, and it is sufficient to describe a Duplicator's winning strategy in $\text{EHR}(G, H, 4, 3)$ for $G, H \in \mathcal{S}$.

In the first round, Spoiler chooses, say, a vertex $v_1 \in V(G)$. By the property (1), there is a vertex $u_1 \in V(H)$ such that in H there is no subgraph W with $u_1 \in V(W)$, $v(W) \leq 21$, $\rho(W, (\{u_1\}, \emptyset)) > 1/\alpha$.

In the second round, Spoiler chooses a vertex $u_2 \in V(H)$. Consider a maximal sequence of graphs $H|_{\{u_1, u_2\}} = H_0 \subset H_1 \subset \dots \subset H_L \subset H$ with each $v(H_i) - v(H_{i-1}) = 1$ and $\rho(H_i, H_{i-1}) > 1/\alpha$. Note that $L \leq 19$. Indeed, if $L > 19$, then

$$\rho(H_{20}, (\{u_1\}, \emptyset)) \geq \frac{40}{21} > \frac{19}{10} > 1/\alpha,$$

that is impossible by the choice of u_1 .

By the choice of u_1 , we have $\rho(H_L, (\{u_1\}, \emptyset)) \leq 1/\alpha$. Moreover, α is not equal to any fraction a/b with $a \leq 20$, hence, the inequality is strict: $\rho(H_L, (\{u_1\}, \emptyset)) < 1/\alpha$. Set $Y = H_L$. By (2), there exists a subgraph X in G such that $Y \cong X$, there is an isomorphism $f : Y \rightarrow X$ such that $f(u_1) = v_1$, and there is no subgraph $W \subset G$ such that $X \subset W$, $v(W) = v(X) + 1$, $\rho(W, X) > 1/\alpha$.

Duplicator chooses $v_2 = f(u_2)$.

In the third round, Spoiler chooses a vertex $v_3 \in V(G)$. Consider two cases.

If $v_3 \in V(X)$, then Duplicator chooses $u_3 = f^{-1}(v_3)$. If after that Spoiler chooses a vertex $u_4 \in V(Y)$, then Duplicator chooses $v_4 = f(u_4)$, and she wins. If Spoiler chooses $u_4 \notin V(Y)$, then $\rho(H|_{\{u_1, u_2, u_3, u_4\}}, H|_{\{u_1, u_2, u_3\}}) < 1/\alpha$. So, the property (2) implies the existence of $v_4 \in V(G)$ such that $v_4 \sim v_i$ if and only if $u_4 \sim u_i$ for all $i \in \{1, 2, 3\}$. Duplicator chooses such a v_4 and wins.

If $v_3 \notin V(X)$, then $\rho(G|_{V(X) \cup \{v_3\}}, X) < 1/\alpha$. So, by (2) there exists a vertex u_3 in H such that there exists an isomorphism $\tilde{f} : G|_{V(X) \cup \{v_3\}} \rightarrow H|_{V(Y) \cup \{u_3\}}$, $\tilde{f}(v_i) = u_i$ for $i \in \{1, 2, 3\}$. Moreover, there is no subgraph $W \subset H$ such that $H|_{V(Y) \cup \{u_3\}} \subset W$, $v(W) = v(Y) + 1$, $\rho(W, Y) > 1/\alpha$. Duplicator chooses that u_3 . Denote $\tilde{X} = G|_{V(X) \cup \{v_3\}}$, $\tilde{Y} = H|_{V(Y) \cup \{u_3\}}$. If in the fourth round Spoiler chooses a vertex $u_4 \in \tilde{Y}$, then Duplicator chooses $\tilde{f}^{-1}(u_4)$ and wins. If Spoiler chooses $u_4 \notin \tilde{Y}$, then $\rho(H|_{\{u_1, u_2, u_3, u_4\}}, H|_{\{u_1, u_2, u_3\}}) < 1/\alpha$. In this case Duplicator's winning strategy is the same as in the first case. \square

Proof of Theorem 5. From Theorem 4, it follows that it is enough to prove that $|S(\phi)| < \infty$ if ϕ is in PNF, $q(\phi) = 4$, $ch(\phi) = 3$.

Let $\phi \in \mathcal{F}$ be a PNF sentence such that $q(\phi) = 4$, $ch(\phi) = 3$. Let $\alpha \in S(\phi)$. Obviously, there exist $\varepsilon > 0$ and sequences n_i, m_i such that, for any $i \in \mathbb{N}$,

$$\min \{P(G(n_i, n_i^{-\alpha}) \models \phi), P(G(m_i, m_i^{-\alpha}) \models \neg(\phi))\} > \varepsilon.$$

By Lemma 6,

$$P(\text{Spoiler has a winning strategy in EHR}(G(n_i, n_i^{-\alpha}), G(m_i, m_i^{-\alpha}), 4, 3)) > \varepsilon^2.$$

Therefore, $\alpha \in \tilde{S}(\phi)$. By Lemma 8, $|\tilde{S}(4) \cap (1/2, 1)| < \infty$. Moreover, the random graph $G(n, n^{-\alpha})$ obeys zero-one 4-law if $\alpha < 1/2$, and the set $S(\phi) \cap (1, \infty)$ is finite (see Section 1). Therefore, $S(\phi) = S(\phi) \cap [1/2, \infty)$ is finite. \square

Note that as the formula (1) with an infinite spectrum is in PNF, Theorem 5 implies that a minimal quantifier depth of a PNF sentence with an infinite spectrum is in $\{5, 6, 7, 8\}$.

Finally, it is easy to see that Lemma 8 and Theorem 4 have a more general corollary which is given below.

Theorem 6 *Let $\phi \in \mathcal{F}$, $q(\phi) = 4$. If either all paths of $F(\phi)$ starting in a root have 3 labels alternations, or all paths of $F(\phi)$ starting in a root have at most 2 labels alternations, then $|S(\phi)| < \infty$.*

From Theorem 6, we get that if there exists a sentence $\phi = \exists x \varphi(x) \in \mathcal{F}$ with $q(\phi) = 4$ and an infinite spectrum, then $F(\phi)$ has both types of paths starting in the root: with maximal number of labels alternations and with less number of labels alternations. So, we still do not have an answer on the first question of Section 1, but we know much more about sentences with $q(\phi) = 4$ and finite spectra.

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